



Traffic on networks: modeling and analysis

Régis Monneau

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Traffic on networks: modeling and analysis

Régis Monneau

(joint work with C. Imbert)

Paris-Est University

Tours; June 27, 2014

Great thanks
for the whole organizing committee !!

A mathematical point of view

$$u_t + H(x, Du, D^2u) = 0$$

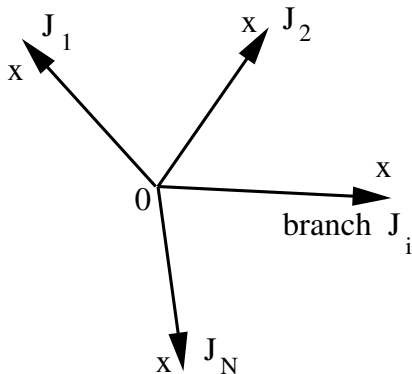
H discontinuous in x

A mathematical point of view

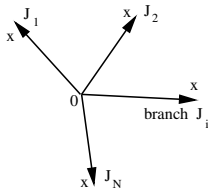
$$u_t + H(x, Du) = 0$$

H discontinuous in x

A junction



Hamilton-Jacobi equations

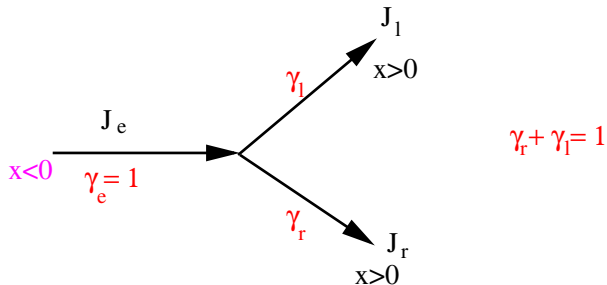


$$\left\{ \begin{array}{ll} u_t^i + H_i(u_x^i) = 0, & x > 0, \quad i = 1, \dots, N \\ u^i = u^j =: u, & x = 0, \\ u_t + F(u_x^1, \dots, u_x^N) = 0, & x = 0 \end{array} \right.$$

- ① Motivation, Hamilton-Jacobi equations and optimal control
- ② Doubling of variables revisited
- ③ General junction conditions
- ④ Homogenization of networks

Motivation

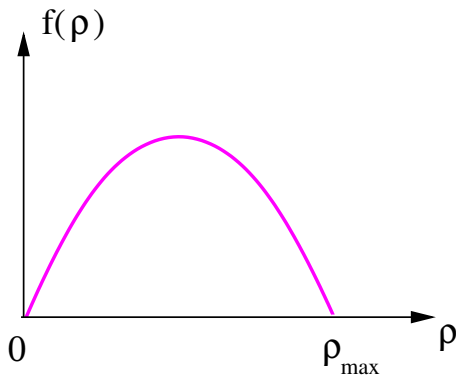
The simple divergent road



LWR model (Lighthill, Whitham 1955 ; Richards 1956) :

$$\rho_t + (f(\rho))_x = 0$$

The fundamental diagram



Some references

Existence for $N \geq 1$

[Garavello, Piccoli (2006)]

Uniqueness result for $N = 2$ branches

[Garavello, Natalini, Piccoli, Terracina (2007)]

[Andreianov, Karlsen, Risebro (2011)]

Uniqueness for $N \geq 1$ for Hamilton-Jacobi equations

[Achdou, Camilli, Cutri, Tchou ; (2013)] : particular Hamiltonians.

[Camilli, Schieborn ; (2013)] : eikonal equations.

[Imbert, M., Zidani (2013)] : particular junction condition

[Imbert, M. preprint (2013-2014)] : general junction conditions

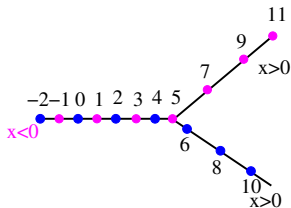
Generalizations

[Barles, Briani, Chasseigne ; (2013), (2014)] : hyperplane (higher dimensions).

[Rao, Zidani ; (2013)], [Rao, Siconolfi, Zidani ; (2013)] : multi-domains.

How to get Hamilton-Jacobi equations ?

$-u^i(x, t)$ = continuous index of the car at (x, t) on the branch i



Getting HJ equations

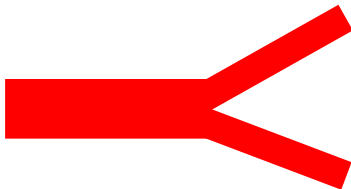
- Take some (space) **primitive** of the density ρ^i with prefactor $1/\gamma_i$
- change some **signs** and the **orientation** of the incoming roads

$$u_t^i + H_i(u_x^i) = 0, \quad x > 0, \quad i = e, l, r$$

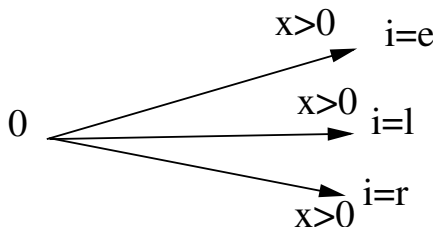
with

$$\left\{ \begin{array}{l} \text{convex } H_i \text{ if } f \text{ concave,} \\ H_i(p) = -\frac{1}{\gamma_i} f(\gamma_i p), \quad i = e \\ H_i(p) = -\frac{1}{\gamma_i} f(-\gamma_i p), \quad i = l, r \end{array} \right.$$

$$f(\rho^i(0^+, t)) = \gamma_i f(\rho^e(0^-, t)) \quad \text{for } i = l, r \quad \text{and} \quad \gamma_e = 1$$



What is the condition at $x = 0$?



$$u_t + F(u_x^e, u_x^l, u_x^r) = 0$$

Optimal control

If

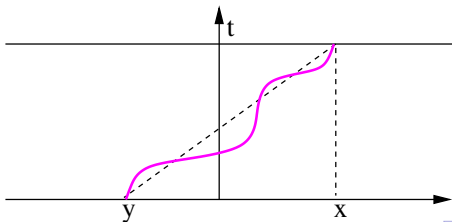
$$\begin{cases} u_t + H(u_x) = 0 & \text{for } x \in \mathbb{R}, \quad t > 0 \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R} \end{cases}$$

with

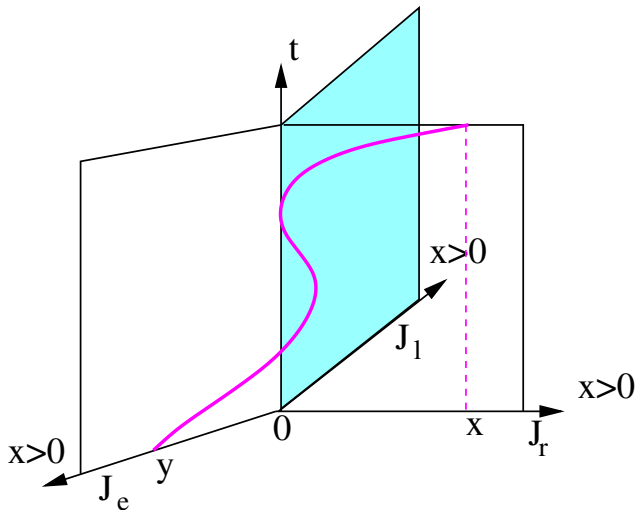
$$H(p) = L^*(p) := \sup_{q \in \mathbb{R}} \{pq - L(q)\}$$

then

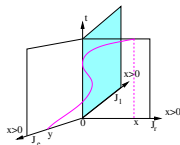
$$u(x, t) = \inf_{\left\{ \begin{array}{l} y \in \mathbb{R}, \\ X(0) = y, \\ X(t) = x \end{array} \right\}} \left\{ u_0(y) + \int_0^t ds \, L(\dot{X}(s)) \right\}$$



Key idea



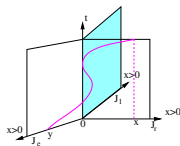
Representation formula



$$J = \bigcup_{i=e,l,r} J_i$$

$$u(x, t) = u_{oc}(x, t) := \inf_{\left\{ \begin{array}{l} y \in J, \\ X(0) = y, \\ X(t) = x \end{array} \right\}} \left\{ u_0(y) + \int_0^t ds \, L(X(s), \dot{X}(s)) \right\}$$

Representation formula



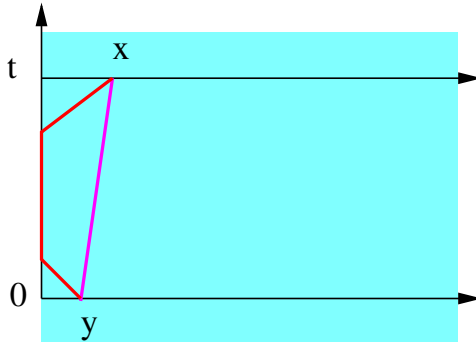
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with a Lagrangian

$$L(x, p) = \begin{cases} L_i(p) & \text{if } x \in J_i \setminus \{0\}, \quad i = e, l, r \\ L_0 & \text{if } x = 0 \end{cases}$$

Analogy with the swimming pool

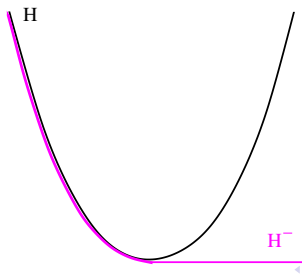


Junction condition with flux limiter A

$$u_t(0, t) + F_A = 0$$

with

$$\left\{ \begin{array}{l} F_A := \max \{A, \max_{i=e,l,r} H_i^-(u_x^i(0^+, t))\} \quad \text{with} \quad A = -L_0 \\ H_i^-(p) = \sup_{q \leq 0} \{pq - L_i(q)\} \end{array} \right.$$

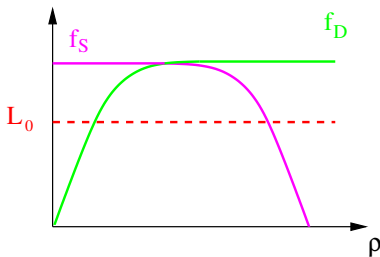


Traffic interpretation of the junction condition

$$u_t(0, t) = \text{passing flux} = \min \left(L_0, \frac{1}{\gamma_e} f_D(\rho^e), \min_{i=l,r} \frac{1}{\gamma_i} f_S(\rho^i) \right)$$

Lebacque condition :

minimum of Demand (f_D) and Supply (f_S)



Riemann Solver of Garavello, Piccoli,

Boundary condition of [Bardos, LeRoux, Nedelec (1979)]

Doubling of variables revisited

The standard doubling of variables method **does not work!**

$$\sup \left\{ u(x, t) - v(y, t) - \frac{(x - y)^2}{\varepsilon} - \dots \right\} \text{ with } x \text{ and } y \text{ on different branches!}$$

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Adaptation with the **vertex test function** G

$$\sup \left\{ u(x, t) - v(y, t) - \varepsilon G \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon} \right) - \dots \right\}$$

$$0 < \eta \leq H(y, -G_y) - H(x, G_x)$$

Contradiction if

$$H(y, -G_y) \leq H(x, G_x)$$

Construction of the test function G

Set of particular solutions for $p = (p^1, \dots, p^N)$

$$u_{p,\lambda}(x, t) = -\lambda t + p^i x \quad \text{on} \quad J_i$$

Germ

$$\mathcal{G}_A = \{(p, \lambda) \quad \text{with} \quad F_A(p) = H_i(p^i) = \lambda \geq A\}$$

Construction of the test function G

Set of particular solutions for $p = (p^1, \dots, p^N)$

$$u_{p,\lambda}(x, t) = -\lambda t + p^i x \quad \text{on} \quad J_i$$

Germ

$$\mathcal{G}_A = \{(p, \lambda) \quad \text{with} \quad F_A(p) = H_i(p^i) = \lambda \geq A\}$$

Let

$$w = w_{p,\lambda}(x, t; y, s) = u_{p,\lambda}(x, t) - u_{p,\lambda}(y, s)$$

which solves

$$H(y, -w_y) = \lambda = H(x, w_x)$$

Construction of the test function G

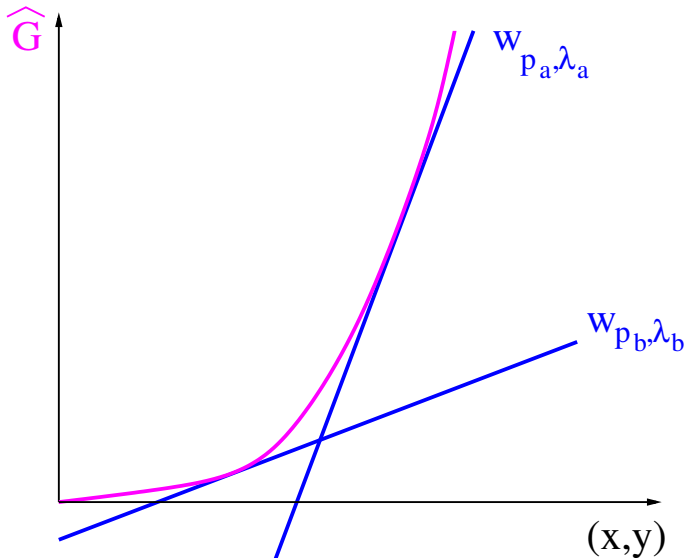
The function

$$\widehat{G}(x, t; y, s) = \sup_{(p, \lambda) \in \mathcal{G}_A} \{w_{p, \lambda}(x, t; y, s)\}$$

satisfies

$$H(y, -\widehat{G}_y) = H(x, \widehat{G}_x) \quad \text{where } \widehat{G} \text{ is smooth}$$

Properties of \hat{G}



Construction of the test function G

Let

$$G^0(x, y) = A + \hat{G}(x, \mathbf{1}; y, \mathbf{0})$$

s.t.

$$G^0(x, y) \geq 0 = G^0(x, x)$$

Construction of the test function G

Let

$$G^0(x, y) = A + \widehat{G}(x, \mathbf{1}; y, \mathbf{0})$$

s.t.

$$G^0(x, y) \geq 0 = G^0(x, x)$$

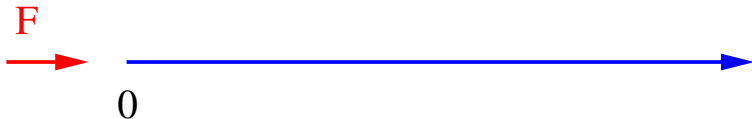
But G^0 is smooth except on the diagonal $\{x = y\}$.

Define $G = G^\delta$ as a smoothing of G^0 on the diagonal such that

$$\begin{cases} G(x, y) \geq 0 & \text{and} & 0 \leq G(x, x) \leq \delta, \\ H(y, -G_y) \leq H(x, G_x) + \delta \end{cases}$$

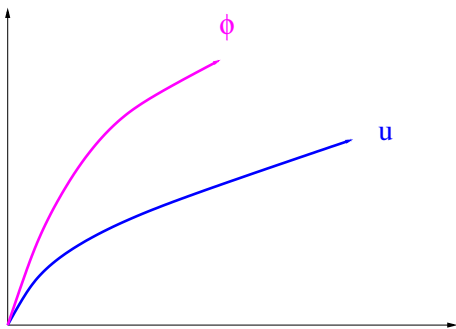
General junction conditions for $N = 1$

Monotonicity of F



$$\begin{cases} u_t + H(u_x) = 0, & x > 0 \\ u_t + F(u_x) = 0, & x = 0 \end{cases}$$

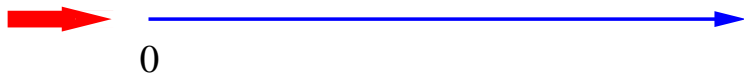
Monotonicity of F



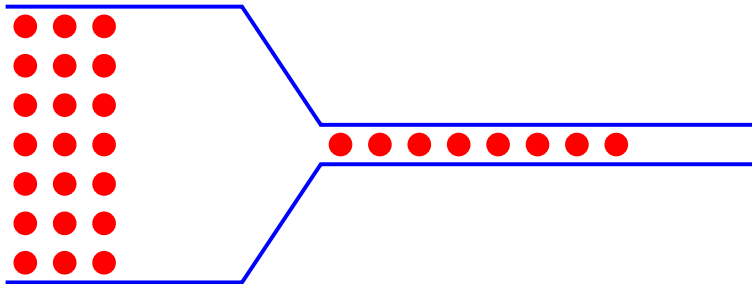
$$\begin{cases} u_t + F(u_x) = 0, & x = 0 \\ \phi_t + F(\phi_x) \leq 0, & x = 0 \end{cases}$$

$F(\rho)$ non increasing in ρ

Meaning of the junction condition



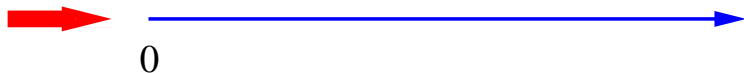
Junction condition **not satisfied** if the imposed flux is **too large**



Relaxation



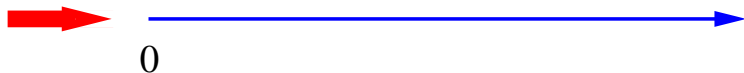
Relaxed junction condition



Relaxed junction condition for subsolutions

$$\left\{ \begin{array}{ll} u_t + F(u_x) \leq 0, & x = 0 \quad (\text{Boundary condition}) \\ \text{OR} \\ u_t + H(u_x) \leq 0, & x = 0 \quad (\text{PDE}) \end{array} \right.$$

Relaxed junction condition



Relaxed junction condition for supersolutions

$$\left\{ \begin{array}{ll} u_t + F(u_x) \geq 0, & x = 0 \quad (\text{Boundary condition}) \\ \text{OR} \\ u_t + H(u_x) \geq 0, & x = 0 \quad (\text{PDE}) \end{array} \right.$$

Theorem (Existence, uniqueness, reduction to F_A -solutions)

If

$$\begin{cases} H \text{ convex and coercive} & (\text{i.e. } H(p) \rightarrow +\infty \text{ as } |p| \rightarrow +\infty) \\ F \text{ continuous and non increasing} \end{cases}$$

Then there exists a unique (viscosity) **relaxed solution** u of

$$\begin{cases} u_t + H(u_x) = 0, & x > 0, \\ u_t + F(u_x) = 0, & x = 0 \end{cases}$$

with Lipschitz initial data

$$u|_{t=0} = u_0$$

Moreover there exists a real A such that u is also solution with F replaced by

$$F_A(u_x) = \max(A, H^-(u_x)),$$

Theorem (Existence, uniqueness, reduction to F_A -solutions)

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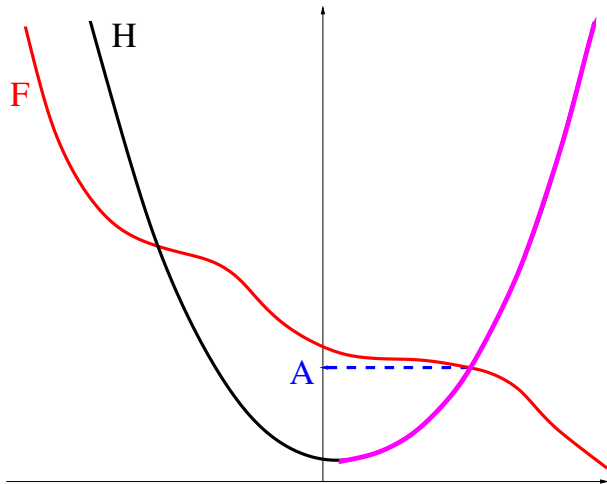
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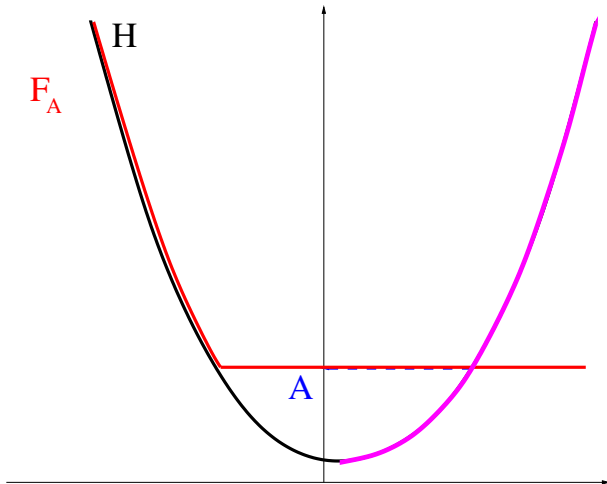
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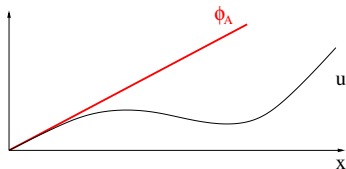






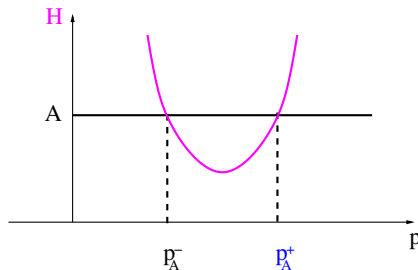
Ramp metering on the A19 in Middlesbrough, UK.

Reduction to a single test function



Check with test functions

$$\varphi(x, t) = \psi(t) + \phi_A(x) \quad \text{with} \quad \phi_A(x) = p_A^+ x$$



Application : reformulation of state constraint problems

For H convex (or quasi-convex) and coercive,

$$\begin{cases} u_t + H(u_x) \geq 0, & x \geq 0, \\ u_t + H(u_x) \leq 0, & x > 0, \end{cases} \quad (\text{state constraint})$$

if and only if

$$\begin{cases} u_t + H(u_x) = 0, & x > 0, \\ u_t + H^-(u_x) = 0, & x = 0, \end{cases} \quad (F_A \text{ condition with } A = -\infty)$$

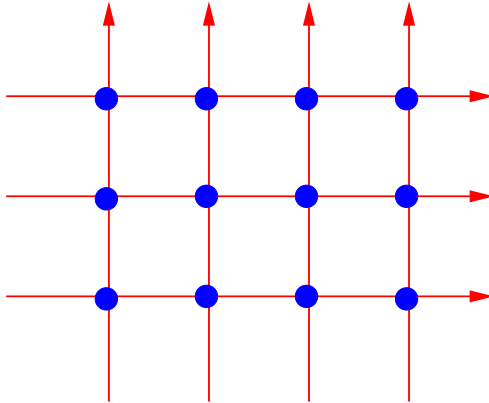
Applications for $N \geq 1$ branches

- relation with Ishii solutions
- characterization of subsolutions on the junction point only
- restriction of solutions on a subdomain
- homogenization
- ...

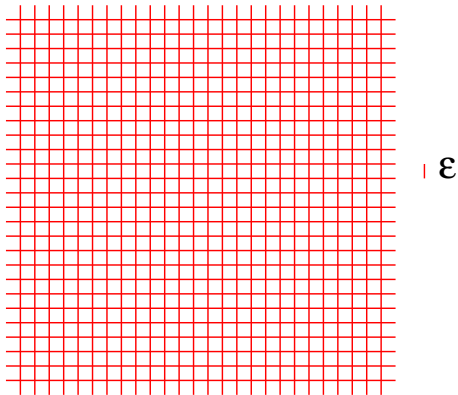
Homogenization of networks

(Numerical simulations done by G. Costeseque)

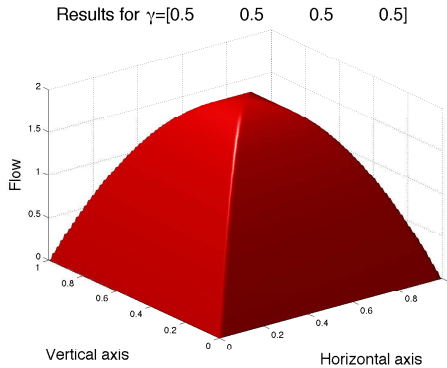
A periodic network



Rescaling of the network

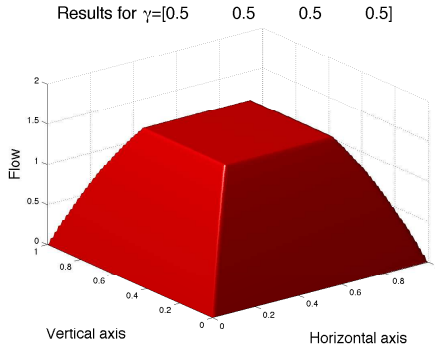


Graph of $-\bar{H}$



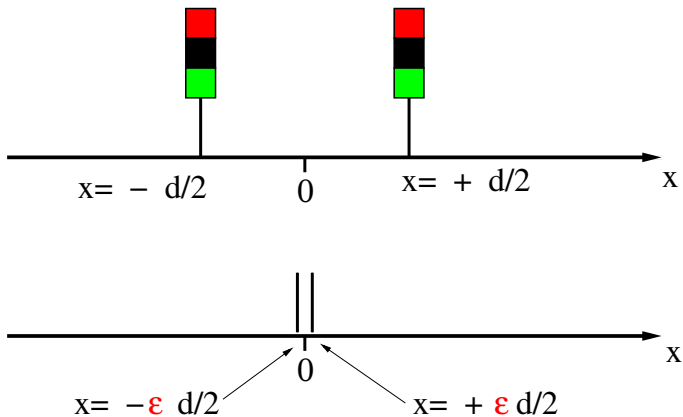
$$\gamma_h = \gamma_v = 1/2, \quad \text{no flux limiter}$$

Graph of $-\bar{H}$



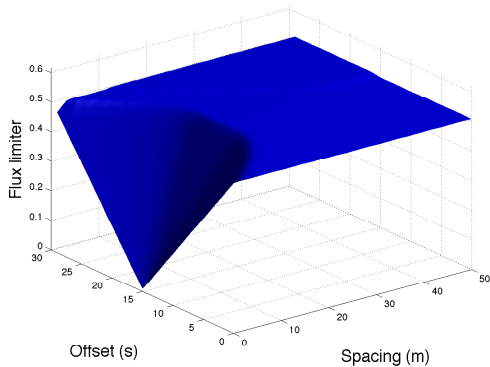
$$\gamma_h = \gamma_v = 1/2, \quad \text{with flux limiter } A$$

Homogenization of traffic lights on a single road



$$-\bar{A}(d, \theta)$$

Flux limiter w.r.t. signals spacing and offset

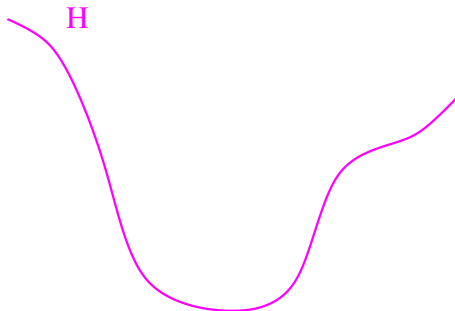


Conclusion :
what we know

What we can do in general

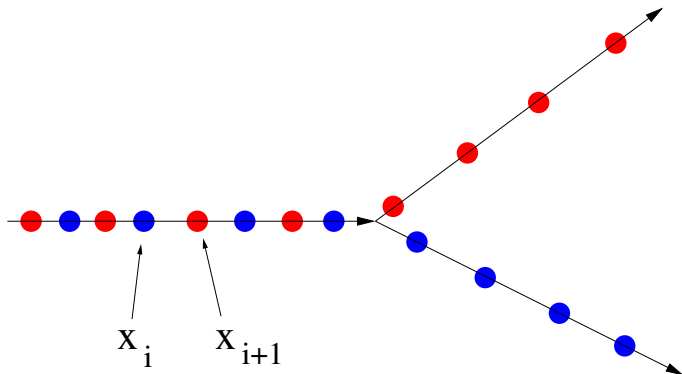
Infinite networks with Hamiltonians $H(x, t, u_x)$ and $F(t, \{u_x^i\}_i)$

Hamiltonians $H(x, t, u_x)$ are convex in u_x , or quasi-convex



Open problems

open problem : micro-macro limit ?



$$\frac{dx_i}{dt} = V_{s(i)}(\{x_j\}_{j \geq i}) \quad \text{with } s(i) = \text{blue or red}$$

weakly related to :

Achdou, Tchou, (preprint : 2014),
Galise, Imbert, M. (preprint 2014)

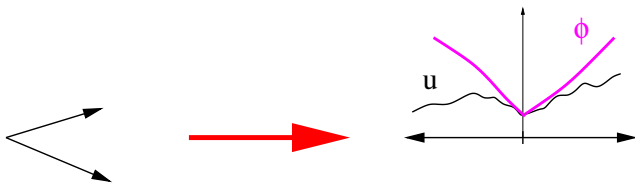
- **traffic** : time evolution/ optimization of the proportions γ_i at each node

- **traffic** : time evolution/ optimization of the proportions γ_i at each node
- more general **non quasi-convex** Hamiltonians in 1D
- general junction conditions in **higher dimensions** :
submanifolds, multi-domains.
- **mixed** first order/second order PDEs

Thank you

The case with $N \geq 1$ branches

Viscosity solution at the junction point



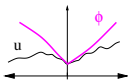
Junction

$$J = \bigcup_{i=1, \dots, N} J_i$$

test function

$$\phi \in C_*^1 \iff \begin{cases} \phi \in C(J \times [0, T)), \\ \phi \in C^1(J_i \times (0, T)), \quad i = 1, \dots, N \end{cases}$$

Viscosity solution at the junction point



Definition (Relaxed subsolution at the junction point)

The USC function u is a **relaxed subsolution** at the junction point, if

$$u \leq \phi \quad \text{with equality at } P_0 = (x_0, t_0) \text{ with } x_0 = 0$$

then at P_0 :

$$\left\{ \begin{array}{l} \phi_t + F(\phi_x^1, \dots, \phi_x^N) \leq 0, \\ \text{OR} \\ \phi_t + H_i(\phi_x^i) \leq 0 \quad \text{for some index } i \end{array} \right.$$

with

$$\phi^i = \phi|_{J_i \times (0, T)}$$

We rewrite

$$\begin{cases} u_t^i + H_i(u_x^i) = 0, & \text{for } (x, t) \in (0, +\infty) \times (0, T), \quad i = 1, \dots, N \\ u_t + F(u_x^1, \dots, u_x^N) = 0, & \text{for } (x, t) \in \{0\} \times (0, T) \end{cases}$$

as

$$u_t + H(x, u_x) = 0 \quad \text{on } J \times (0, T) \quad \text{with } J = \bigcup_{i=1, \dots, N} J_i \quad (1)$$

with

$$H(x, p) = \begin{cases} H_i(p) & \text{convex coercive} & \text{if } x \in J_i \setminus \{0\} \\ F(p^1, \dots, p^N) & \text{non increasing} & \text{if } x = 0 \end{cases}$$

The PDE

$$u_t + H(x, u_x) = 0 \quad \text{on} \quad J \times (0, T) \quad (1)$$

Initial condition

$$u(x, 0) = u_0(x) \quad \text{for all} \quad x \in J, \quad \text{with} \quad u_0 \in \text{Lip}(J) \quad (2)$$

The PDE

$$u_t + H(x, u_x) = 0 \quad \text{on} \quad J \times (0, T) \quad (1)$$

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There exists a unique (viscosity) *relaxed solution* u of (1)-(2).

Moreover there exists a real A such that u is also solution with F replaced by

$$F_A(u_x^1, \dots, u_x^N) = \max(A, \max_{i=1, \dots, N} H_i^-(u_x^i))$$

where

$$A = F((H_1^+)^{-1}(A), \dots, (H_N^+)^{-1}(A)) \quad \text{or} \quad A = A_0 := \max_{i=1, \dots, N} \left(\min_{\mathbb{R}} H_i \right)$$